Malaysian Journal of Mathematical Sciences 20(1): 43–55 (2026)
https://doi.org/10.47836/mjms.20.1.03

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Properties of Max-Plus Algebraic Determinants Derived from Duality Theorem

Nishida, Y. *[1], Watanabe, S. [2], and Watanabe, Y.[3]

[1]*Department of Science, Technology and Informatics, Kyoto Prefectural University, Japan*
[2]*Department of Informatics, The University of Fukuchiyama, Japan*
[3]*Applied Ultrasonic Research Center, Doshisha University, Japan*

E-mail: y-nishida@kpu.ac.jp
Corresponding author

Received: 7 May 2025
Accepted: 9 July 2025

Abstract

Max-plus algebra is a semiring with two operations: addition $\oplus := \max$ and multiplication $\otimes := +$. The definition of the max-plus algebraic determinant is equivalent to the assignment problem on a bipartite graph. Since the assignment problem has a formulation as linear programming, the duality theorem induces the minimization problem that attains the same optimal value as the assignment problem. We translate this dual problem in terms of max-plus algebraic operation and obtain another expression for the max-plus algebraic determinant. For the determinant of the product of max-plus square matrices, we have only the inequality $\det(P \otimes Q) \geq \det P \otimes \det Q$, and a known sufficient condition for the equality is given by the condition for $\det(P \otimes Q)$. Exploiting the duality theorem for the determinant, we derive a necessary and sufficient condition for the equality. Our criterion only needs the optimal assignments corresponding to $\det P$ and $\det Q$ but does not require to compute $\det(P \otimes Q)$ beforehand.

Keywords: max-plus algebra; duality theorem; tropical semiring; determinant.

1 Introduction

Max-plus algebra is a commutative semiring associated with the addition $\oplus := \max$ and the multiplication $\otimes := +$. It has been applied to discrete event systems such as timed event graphs, heap models, manufacturings, rail systems, network calculus, and so on. Min-plus algebra, which is essentially the same as max-plus algebra, has been developed in connection with the shortest path problem. Moreover, max-plus algebra is known as an underlying algebra of tropical geometry.

In this paper, we focus on determinants of max-plus square matrices. In max-plus algebra, the "signs" of numbers are not considered because there is no inverse element for addition. Hence, determinants of matrices are defined as in the usual algebra by omitting the signs of permutations, i.e., they are the same as permanents. From the viewpoint of combinatorial optimization, computing the max-plus algebraic determinant is known to be equivalent to solving the problem of finding a perfect matching with the maximum weight on a bipartite graph, which is called the optimal assignment problem. That problem for an n-by-n max-plus matrix with m finite entries can be solved by the Hungarian method [14] in $O(n(m + n \log n))$ time of computation when the improvement proposed in [17] is applied.

Several kinds of other algorithms for the optimal assignment problem, such as primal algorithms, dual algorithms, and the Dinic-Kronrod algorithm, are summarized in [3]. The assignment problem can be expressed as an integer programming. Due to the total unimodularity of the coefficient matrix, the duality theorem shows that the optimal value of the assignment problem coincides with the dual one. The constraints of the dual of the assignment problem are defined by the transpose of the incident matrix of the bipartite graph. Such kind of system of inequalities is studied in [4] as the system of dual network inequalities. Tropical geometric characterization of the dual feasible solution set is given in [12]. The diagonal scaling of matrices establishes a characterization of dual optimal solutions in terms of the eigenvalue problem [10].

In usual linear algebra, a famous identity on determinants of matrices is $\det(PQ) = \det P \det Q$. Surprisingly, a max-plus analogue of this identity does not hold, while the inequality,

$$\det(P \otimes Q) \geq \det P \otimes \det Q,$$

still holds for any square max-plus matrices P and Q of the same sizes. (Note that the symbol $\det$ here and hereafter denotes the max-plus algebraic determinant.) This inequality was mentioned in [15] and also derived in [2] as a corollary of the max-plus Cauchy-Binet formula [7]. A sufficient condition for P and Q to satisfy the equality,

$$\det(P \otimes Q) = \det P \otimes \det Q, \tag{1}$$

was proposed in [15] by considering the parities of the permutations attaining $\det(P \otimes Q)$. An inequality for bideterminants of max-plus matrices presented in [9] also implies a sufficient condition for (1). Similar inequalities are found in the symmetrized max-plus algebra [8] and the supertropical algebra [11].

In Section 2, a new expression of the max-plus algebraic determinant, say, the dual-type formula, is presented. We interpret the duality theorem for the assignment problem into the max-plus algebraic framework. Since the determinant is identical to the optimal value of the assignment problem, it is also achieved as that of the dual one. More precisely, if we denote the usual arithmetic sum of all entries of vector $\boldsymbol{z}$ by $\nu(\boldsymbol{z})$, the determinant of a matrix P coincides with the minimum value of $\nu(P \otimes \boldsymbol{z})$ divided by $\nu(\boldsymbol{z})$ in the context of max-plus algebra.

Section 3 is the discussion on (1). Our contribution in this paper is to derive an equivalent condition for P and Q to satisfy (1), while only sufficient conditions have been considered so far. We reduce the validity of (1) to the max-plus algebraic eigenvalue problem by exploiting the dual-type formula of max-plus algebraic determinants. Although the very restrictive condition that requires the invertibility of matrices was derived in [1], other known sufficient conditions are described in terms of the product $P \otimes Q$. On the other hand, the criterion presented in this paper uses only knowledge of $\det P$ and $\det Q$, not of $\det(P \otimes Q)$. This enables us to check whether (1) holds in $O(n(m + n \log n))$ time of computation, where n and m represent the order and the total number of finite entries of matrices, respectively.

2 Duality Theorem for Determinants of Max-Plus Matrices

Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ be the real numbers $\mathbb{R}$ augmented by $-\infty$. For $a, b \in \mathbb{R}_{\max}$, addition $\oplus$ and multiplication $\otimes$ are respectively defined by:

$$a \oplus b := \max(a, b), \quad a \otimes b := a + b.$$

Then, $(\mathbb{R}_{\max}, \oplus, \otimes)$ is max-plus algebra. The additive identity is $\varepsilon := -\infty$ and the multiplicative identity is $e := 0$. The inverse of $a \in \mathbb{R}$ cannot be defined for addition $\oplus$, while the max-plus division is $a \oslash b := a \otimes (-b)$ for $a \in \mathbb{R}_{\max}$ and $b \in \mathbb{R}$.

Let $\mathbb{R}^n_{\max}$ and $\mathbb{R}^{m\times n}_{\max}$ denote the sets of all n-dimensional vectors and m-by-n matrices. The zero vector is $\mathcal{E}_n := (\varepsilon, \varepsilon, \ldots, \varepsilon)^\top \in \mathbb{R}^n_{\max}$. For $P, Q \in \mathbb{R}^{m\times n}_{\max}$, the matrix sum $P \oplus Q \in \mathbb{R}^{m\times n}_{\max}$ is

$$[P \oplus Q]_{ij} := [P]_{ij} \oplus [Q]_{ij}.$$

Here, $[P]_{ij}$ stands for the (i, j) entry of the matrix P. For $P \in \mathbb{R}^{l\times m}_{\max}$ and $Q \in \mathbb{R}^{m\times n}_{\max}$, the matrix product $P \otimes Q \in \mathbb{R}^{l\times n}_{\max}$ is

$$[P \otimes Q]_{ij} := \bigoplus_{k=1}^{m} [P]_{ik} \otimes [Q]_{kj}.$$

For $P \in \mathbb{R}^{m\times n}_{\max}$ and $\alpha \in \mathbb{R}_{\max}$, the scalar multiplication $\alpha \otimes P \in \mathbb{R}^{m\times n}_{\max}$ is

$$[\alpha \otimes P]_{ij} := \alpha \otimes [P]_{ij}.$$

A matrix $P \in \mathbb{R}^{n\times n}_{\max}$ has its inverse $P^{\otimes -1}$ if and only if it contains exactly one finite entry in each row and column. Such a matrix is called a generalized permutation matrix.

The determinant of $P = (p_{ij}) \in \mathbb{R}^{n\times n}_{\max}$ is defined by:

$$\det P = \bigoplus_{\sigma \in S_n} \bigotimes_{i=1}^{n} p_{i\sigma(i)} = \max_{\sigma \in S_n} \sum_{i=1}^{n} p_{i\sigma(i)}, \tag{2}$$

where S_n is the set of permutation of $\{1, 2, \ldots, n\}$. The set of permutations attaining the maximum in (2) is denoted by $S(P)$. Computing the max-plus algebraic determinant is equivalent to solving the assignment problem. Let us consider the bipartite graph with the disjoint vertex sets $\{l_1, l_2, \ldots, l_n\}$ and $\{r_1, r_2, \ldots, r_n\}$, and the edge set $\{(l_i, r_j) \mid p_{ij} \neq \varepsilon\}$. The edge cost is given by $c((l_i, r_j)) = p_{ij}$. The assignment problem in this bipartite graph is to find a perfect matching, that is, the subset of edges covering every vertex exactly once, so that the total edge cost is maximized.

The bipartite graph has no perfect matching if and only if $\det P = \varepsilon$. The assignment problem is expressed as an integer linear programming:

$$\begin{aligned} &\text{maximize} && \sum_{i,j=1}^{n} p_{ij} x_{ij}, \\ &\text{subject to} && \sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1, 2, \dots, n, \\ &&& \sum_{i=1}^{n} x_{ij} = 1, \qquad j = 1, 2, \dots, n, \\ &&& x_{ij} \in \{0, 1\}, \qquad i, j = 1, 2, \dots, n. \end{aligned} \tag{P}$$

The linear relaxation of (P) is given by replacing all constraints $x_{ij} \in \{0, 1\}$ with $0 \le x_{ij} \le 1$. As the first two constraints of (P) induce $x_{ij} \le 1$ for all i and j, the linear relaxation is equivalent to the problem (LP) below:

$$\begin{aligned} &\text{maximize} && \sum_{i,j=1}^{n} p_{ij} x_{ij}, \\ &\text{subject to} && \sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1, 2, \dots, n, \\ &&& \sum_{i=1}^{n} x_{ij} = 1, \qquad j = 1, 2, \dots, n, \\ &&& x_{ij} \ge 0, \qquad i, j = 1, 2, \dots, n. \end{aligned} \tag{LP}$$

It is well known that the optimal values of (P) and (LP) coincide since the coefficient matrix, which is the incidence matrix of a bipartite graph, is totally unimodular, i.e., all minors are $-1, 0$ or 1 [16]. The dual problem of (LP) is as follows:

$$\begin{aligned} &\text{minimize} && \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j, \\ &\text{subject to} && u_i + v_j \ge p_{ij}, \qquad i, j = 1, 2, \dots, n. \end{aligned} \tag{D}$$

Note that the constraints of (D) are also valid for the case with $p_{ij} = \varepsilon$. By the duality theorem [16], the optimal values of (LP) and (D) coincide, if exist.

Now, we focus on the constraints of the problem (D). The inequalities for $j = 1, 2, \dots, n$ are integrated as:

$$u_i \ge \bigoplus_{j=1}^{n} p_{ij} \otimes (-v_j), \qquad i = 1, 2, \dots, n. \tag{3}$$

Let $\boldsymbol{u} = (u_1, u_2, \dots, u_n)^\top$ and $\boldsymbol{v} = (v_1, v_2, \dots, v_n)^\top$. Then, (3) can be represented in the matrix form:

$$\boldsymbol{u} \ge P \otimes (-\boldsymbol{v}). \tag{4}$$

Since we are considering the minimization of the objective function, each variable u_i should be as small as possible. Hence, for the optimal solution, the inequality in (4) can be replaced with the equality, that is

$$\boldsymbol{u} = P \otimes (-\boldsymbol{v}).$$

For $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)^\top \in \mathbb{R}^n_{\max}$, let,

$$\nu(\boldsymbol{u}) = u_1 + u_2 + \ldots + u_n.$$

Then, the problem (D) is expressed as the unconstrained optimization problem,

$$\text{minimize} \qquad \nu(P \otimes (-\boldsymbol{v})) \otimes \nu(\boldsymbol{v}).$$

Thus, if (P) has a feasible solution, then we have

$$\det P = \min_{\boldsymbol{v} \in \mathbb{R}^n} \nu(P \otimes (-\boldsymbol{v})) \otimes \nu(\boldsymbol{v}).$$

If (P) is not feasible, we see that the objective function of (D) has no lower bound by the duality theorem. Hence,

$$\det P = \min_{\boldsymbol{v} \in \mathbb{R}^n} \nu(P \otimes (-\boldsymbol{v})) \otimes \nu(\boldsymbol{v}) = \varepsilon.$$

Replacing $\boldsymbol{v} \in \mathbb{R}^n$ with $-\boldsymbol{z}$, we have the first main theorem.

Theorem 2.1. *Let* $P \in \mathbb{R}^{n \times n}_{\max}$. *Then,*

$$\det P = \min_{\boldsymbol{z} \in \mathbb{R}^n} \nu(P \otimes \boldsymbol{z}) \oslash \nu(\boldsymbol{z}).$$

Suppose $\det P \neq \varepsilon$. We next characterize the set,

$$d(P) := \{\boldsymbol{z} \in \mathbb{R}^n \mid \nu(P \otimes \boldsymbol{z}) \oslash \nu(\boldsymbol{z}) = \det P\},$$

in terms of the eigenvalue problem. Similar approaches can be found in max-plus matrix scaling [10]. Here, we present a short introduction to the max-plus eigenvalue problem [6].

For $P \in \mathbb{R}^{n \times n}_{\max}$, we define $\lambda \in \mathbb{R}_{\max}$ as an eigenvalue of P if some vector $\boldsymbol{x} \in \mathbb{R}^n_{\max} \setminus \{\mathcal{E}_n\}$, namely, an eigenvector, satisfies,

$$P \otimes \boldsymbol{x} = \lambda \otimes \boldsymbol{x}.$$

The maximum eigenvalue of $P = (p_{ij}) \in \mathbb{R}^{n \times n}_{\max}$ can be characterized by the associated digraph $\mathcal{G}(P)$. The vertex set of $\mathcal{G}(P)$ is

$$V = \{1, 2, \ldots, n\},$$

and the edge set is

$$E = \{(i, j) \mid p_{ij} \neq \varepsilon\},$$

with weight $w((i, j)) = p_{ij}$. A path in $\mathcal{G}(P)$ is a sequence $(i_0, i_1, \ldots, i_\ell)$ of vertices where $(i_{k-1}, i_k) \in E$ for $k = 1, 2, \ldots, \ell$. It is called a circuit if $i_0 = i_\ell$. The number of edges contained in a path $\mathcal{P} = (i_0, i_1, \ldots, i_\ell)$ is called the length $\ell(\mathcal{P})$ of $\mathcal{P}$. The sum,

$$w(\mathcal{P}) := \sum_{k=1}^{\ell} w((i_{k-1}, i_k)),$$

is the weight of $\mathcal{P}$. The average circuit weight of a circuit $\mathcal{C}$ is the ratio of its weight to its length, namely:

$$\text{ave}(\mathcal{C}) = \frac{w(\mathcal{C})}{\ell(\mathcal{C})}.$$

If $\mathcal{G}(P)$ contains no positive weight circuit, then the (i, j) entry of

$$P^+ := P \oplus P^{\otimes 2} \oplus \ldots \oplus P^{\otimes n},$$

stands for the maximum weight of all paths from vertex i to j in $\mathcal{G}(P)$. The maximum average circuit weight in the associated digraph $\mathcal{G}(P)$ is equal to the maximum eigenvalue of P. The eigenspace with respect to the eigenvalue λ is generated by the column vectors of $((-\lambda) \otimes P)^+$ whose diagonal entries are 0.

Let $W(P) = \{\boldsymbol{x} \in \mathbb{R}^n \mid P \otimes \boldsymbol{x} = \boldsymbol{x}\}$. For a matrix $P = (p_{ij}) \in \mathbb{R}_{\max}^{n\times n}$ and a permutation $\sigma \in S_n$, the matrix P_σ is defined by:

$$[P_\sigma]_{ij} = \begin{cases} [P]_{ij}, & \text{if } j = \sigma(i), \\ \varepsilon, & \text{otherwise.} \end{cases}$$

If $p_{i\sigma(i)} \neq \varepsilon$ for $i = 1, 2, \ldots, n$, then we can easily verify that P_σ is a generalized permutation matrix.

Proposition 2.1. *Let $P \in \mathbb{R}_{\max}^{n\times n}$. If $\det P \neq \varepsilon$, then we have*

$$d(P) = W(P_\sigma^{\otimes -1} \otimes P),$$

for any $\sigma \in S(P)$.

Proof. Take any $\sigma \in S(P)$. Recall that $\boldsymbol{z} = (z_1, z_2, \ldots, z_n)^\top \in d(P)$, means that $(\boldsymbol{u}, \boldsymbol{v}) = (P \otimes \boldsymbol{z}, -\boldsymbol{z})$ is an optimal solution to (D). We have

$$-u_i + p_{ik} - v_k \leq 0,$$

for all $i, k \in \{1, 2, \ldots, n\}$ by the constraints of (D). In particular, since $x_{i\sigma(i)} = 1 > 0$ for the primal optimal solution $(x_{ij}) \in \mathbb{R}^{n\times n}$, we see by the complementarity condition that,

$$-u_i + p_{i\sigma(i)} - v_{\sigma(i)} = 0,$$

for all $i \in \{1, 2, \ldots, n\}$. Let $D(\boldsymbol{z}) \in \mathbb{R}_{\max}^{n\times n}$ be the diagonal matrix with $z_1, z_2, \ldots, z_n$ on its diagonal. All entries of the matrix,

$$D(-(P \otimes \boldsymbol{z})) \otimes P \otimes D(\boldsymbol{z}), \tag{5}$$

are nonpositive, and in particular, $(i, \sigma(i))$ entries are 0 for all i, because the (i, k) entry of (5) is $-u_i + p_{ik} - v_k$. On the other hand, the (i, k) entry of (5) is computed as

$$-\left(\bigoplus_{j=1}^{n} p_{ij} \otimes z_j\right) \otimes p_{ik} \otimes z_k \leq 0.$$

Hence, we see that $\boldsymbol{z} \in d(P)$ if and only if,

$$\bigoplus_{j=1}^{n} p_{ij} \otimes z_j = p_{i\sigma(i)} \otimes z_{\sigma(i)},$$

for $i = 1, 2, \ldots, n$, which implies $P \otimes \boldsymbol{z} = P_\sigma \otimes \boldsymbol{z}$. Thus, $\boldsymbol{z} \in d(P)$ is equivalent to

$$\boldsymbol{z} \in W(P_\sigma^{\otimes -1} \otimes P).$$

This proves the proposition. □

Remark 2.1. *Since all diagonal entries of* $P_\sigma^{\otimes -1} \otimes P$ *are* 0 *and the associated digraph* $\mathcal{G}(P_\sigma^{\otimes -1} \otimes P)$ *has no positive weight circuit, the entries of* $(P_\sigma^{\otimes -1} \otimes P)^{\otimes k}$ *stand for the maximum weights of paths in* $\mathcal{G}(P_\sigma^{\otimes -1} \otimes P)$ *whose length is up to* k*. In particular, we have*

$$(P_\sigma^{\otimes -1} \otimes P)^+ = (P_\sigma^{\otimes -1} \otimes P)^{\otimes k},$$

for any $k \geq n$*, and*

$$W(P_\sigma^{\otimes -1} \otimes P) = \{(P_\sigma^{\otimes -1} \otimes P)^+ \otimes \boldsymbol{u} \mid \boldsymbol{u} \in \mathbb{R}^n\},$$

by Theorem 1.6.18 of [6].

Example 2.1. *Consider a matrix,*

$$P = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}.$$

The max-plus algebraic determinant of P *is*

$$\det P = 3 \otimes 2 \oplus 0 \otimes 1 = 5,$$

which is attained by $\sigma := \mathrm{id} \in S(P)$*. The dual optimal solutions are the finite eigenvectors of*

$$P_\sigma^{\otimes -1} \otimes P = \begin{pmatrix} -3 & \varepsilon \\ \varepsilon & -2 \end{pmatrix} \otimes \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix}.$$

Since we have $(P_\sigma^{\otimes -1} \otimes P)^+ = P_\sigma^{\otimes -1} \otimes P$ *in this example, the dual optimal solutions are the max-plus algebraic linear combinations of* $\boldsymbol{z}_1 := \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ *and* $\boldsymbol{z}_2 := \begin{pmatrix} -3 \\ 0 \end{pmatrix}$*. For example, by taking,*

$$\boldsymbol{z} := \boldsymbol{z}_1 \oplus 1 \otimes \boldsymbol{z}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we observe that,

$$\nu(P \otimes \boldsymbol{z}) \oslash \nu(\boldsymbol{z}) = ((3 \oplus 1) + (1 \oplus 3)) \oslash (0 + 1) = 5 = \det P.$$

The set of dual optimal solutions is illustrated in Figure 1.

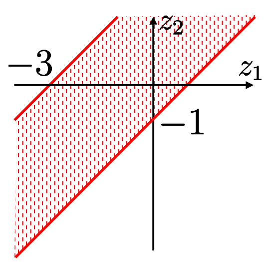

Figure 1: Dual optimal solutions for P in Example 2.1.

3 Determinants of Matrix Product

In the usual algebra, the identity $\det(PQ) = \det P \det Q$ holds for any matrices $P, Q \in \mathbb{R}^{n\times n}$. However, the matrices below provide a counterexample for the max-plus analogue of this identity.

Example 3.1. *Consider two matrices,*

$$P = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 2 \\ 5 & 6 \end{pmatrix}.$$

The determinants of these matrices are $\det P = 5$ *and* $\det Q = 7$*, respectively. Hence,*

$$\det(P \otimes Q) = \det \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = 13 \neq 12 = \det P \otimes \det Q.$$

It is shown in literature that the inequality,

$$\det(P \otimes Q) \geq \det P \otimes \det Q, \tag{6}$$

always holds for any $P, Q \in \mathbb{R}^{n\times n}_{\max}$. A sufficient condition for the equality in (6) is presented in [15].

Proposition 3.1 ([15])**.** *Let* $P, Q \in \mathbb{R}^{n\times n}_{\max}$*. The equality,*

$$\det(P \otimes Q) = \det P \otimes \det Q, \tag{7}$$

holds if $S(P \otimes Q)$ *does not have two permutations in different parity.*

In this section, we aim at deriving an equivalent condition for P and Q to satisfy (7).

Theorem 3.1. *Let* $P, Q \in \mathbb{R}^{n\times n}_{\max}$ *and suppose* $\det P \neq \varepsilon$ *and* $\det Q \neq \varepsilon$*. Take any permutations* $\sigma \in S(P)$ *and* $\pi \in S(Q)$*. Then, (7) is satisfied if and only if,*

$$W(P_\sigma^{\otimes -1} \otimes P) \cap W(Q \otimes Q_\pi^{\otimes -1}) \neq \emptyset.$$

Proof. "Only if" part: Take a vector $\boldsymbol{z}^* \in d(P \otimes Q)$. By Theorem 2.1, we have

$$\begin{aligned} \det(P \otimes Q) &= \nu(P \otimes Q \otimes \boldsymbol{z}^*) \oslash \nu(\boldsymbol{z}^*) \\ &= (\nu(P \otimes Q \otimes \boldsymbol{z}^*) \oslash \nu(Q \otimes \boldsymbol{z}^*)) \otimes (\nu(Q \otimes \boldsymbol{z}^*) \oslash \nu(\boldsymbol{z}^*)). \end{aligned}$$

Moreover, by applying Theorem 2.1 to P and Q, we have

$$\det P \leq \nu(P \otimes Q \otimes \boldsymbol{z}^*) \oslash \nu(Q \otimes \boldsymbol{z}^*), \tag{8}$$

and

$$\det Q \leq \nu(Q \otimes \boldsymbol{z}^*) \oslash \nu(\boldsymbol{z}^*). \tag{9}$$

Combining them, we obtain,

$$\det(P \otimes Q) = (\nu(P \otimes Q \otimes \boldsymbol{z}^*) \oslash \nu(Q \otimes \boldsymbol{z}^*)) \otimes (\nu(Q \otimes \boldsymbol{z}^*) \oslash \nu(\boldsymbol{z}^*)) \geq \det P \otimes \det Q.$$

Thus, when we assume that (7) holds true, considering (8) and (9), we must have

$$\det P = \nu(P \otimes Q \otimes \boldsymbol{z}^*) \oslash \nu(Q \otimes \boldsymbol{z}^*),$$

and

$$\det Q = \nu(Q \otimes \boldsymbol{z}^*) \oslash \nu(\boldsymbol{z}^*).$$

By Proposition 2.1, these equalities mean that,

$$Q \otimes \boldsymbol{z}^* \in W(P_\sigma^{\otimes -1} \otimes P),$$

and

$$\boldsymbol{z}^* \in W(Q_\pi^{\otimes -1} \otimes Q),$$

respectively. The latter leads to,

$$(Q \otimes Q_\pi^{\otimes -1}) \otimes (Q \otimes \boldsymbol{z}^*) = Q \otimes (Q_\pi^{\otimes -1} \otimes Q) \otimes \boldsymbol{z}^* = Q \otimes \boldsymbol{z}^*.$$

Hence, we have

$$Q \otimes \boldsymbol{z}^* \in W(Q \otimes Q_\pi^{\otimes -1}).$$

Thus, we obtain,

$$W(P_\sigma^{\otimes -1} \otimes P) \cap W(Q \otimes Q_\pi^{\otimes -1}) \neq \emptyset,$$

proving the "only if" part of the theorem.

"If" part: Suppose $W(P_\sigma^{\otimes -1} \otimes P) \cap W(Q \otimes Q_\pi^{\otimes -1}) \neq \emptyset$. Then, take a vector,

$$\boldsymbol{y}^* \in W(P_\sigma^{\otimes -1} \otimes P) \cap W(Q \otimes Q_\pi^{\otimes -1}).$$

Considering Remark 2.1, we can take $\boldsymbol{u} \in \mathbb{R}^n$ such that,

$$\boldsymbol{y}^* = (Q \otimes Q_\pi^{\otimes -1})^{\otimes (n+1)} \otimes \boldsymbol{u}.$$

Setting,

$$\boldsymbol{z}^* = (Q_\pi^{\otimes -1} \otimes Q)^{\otimes n} \otimes Q_\pi^{\otimes -1} \otimes \boldsymbol{u},$$

we have

$$(Q_\pi^{\otimes -1} \otimes Q) \otimes \boldsymbol{z}^* = \boldsymbol{z}^*,$$

by using the fact that $(Q_\pi^{\otimes -1} \otimes Q)^{\otimes (n+1)} = (Q_\pi^{\otimes -1} \otimes Q)^{\otimes n}$. Hence, we obtain the pair $(\boldsymbol{y}^*, \boldsymbol{z}^*)$ such that $\boldsymbol{y}^* \in d(P)$, $\boldsymbol{z}^* \in d(Q)$, and $\boldsymbol{y}^* = Q \otimes \boldsymbol{z}^*$. Then, we have

$$\begin{aligned}
\det(P \otimes Q) &= \min_{\boldsymbol{z} \in \mathbb{R}^n} \nu(P \otimes Q \otimes \boldsymbol{z}) \oslash \nu(\boldsymbol{z}) \\
&\leq \nu(P \otimes Q \otimes \boldsymbol{z}^*) \oslash \nu(\boldsymbol{z}^*) \\
&= (\nu(P \otimes \boldsymbol{y}^*) \oslash \nu(\boldsymbol{y}^*)) \otimes (\nu(Q \otimes \boldsymbol{z}^*) \oslash \nu(\boldsymbol{z}^*)) \\
&= \det P \otimes \det Q.
\end{aligned}$$

Since $\det(P \otimes Q) \geq \det P \otimes \det Q$ always holds, we obtain (7). □

The condition that $W(P_\sigma^{\otimes -1} \otimes P) \cap W(Q \otimes Q_\pi^{\otimes -1}) \neq \emptyset$ can be checked easily.

Corollary 3.1. *Let $P, Q \in \mathbb{R}_{\max}^{n\times n}$ and suppose $\det P \neq \varepsilon$ and $\det Q \neq \varepsilon$. Take any permutations $\sigma \in S(P)$ and $\pi \in S(Q)$. Then, (7) is satisfied if and only if the digraph $\mathcal{G}(P_\sigma^{-1} \otimes P \oplus Q \otimes Q_\pi^{-1})$ has no circuit with positive weight.*

We use the following result for the proof of the above corollary.

Lemma 3.1 (Corollary 9 of Butkovič (1985) [5])**.** *Let $P, Q \in \mathbb{R}_{\max}^{n\times n}$. If all diagonal entries of P and Q are nonnegative, then we have*

$$W(P) \cap W(Q) = W(P \oplus Q).$$

Proof of Corollary 3.1. By Theorem 3.1, (7) is satisfied if and only if,

$$W(P_\sigma^{-1} \otimes P) \cap W(Q \otimes Q_\pi^{-1}) \neq \emptyset.$$

Applying Lemma 3.1 to $P_\sigma^{-1} \otimes P$ and $Q \otimes Q_\pi^{-1}$, we have

$$W(P_\sigma^{-1} \otimes P) \cap W(Q \otimes Q_\pi^{-1}) = W(P_\sigma^{-1} \otimes P \oplus Q \otimes Q_\pi^{-1}).$$

Since the eigenvalue corresponding to a finite eigenvector is equal to the maximum average circuit weight in the associated digraph, we see that $W(P_\sigma^{-1} \otimes P \oplus Q \otimes Q_\pi^{-1}) \neq \emptyset$ if and only if $\mathcal{G}(P_\sigma^{-1} \otimes P \oplus Q \otimes Q_\pi^{-1})$ has no positive weight circuit. □

Example 3.2. *Consider the matrices P and Q in Example 3.1. Take $\sigma = \mathrm{id} \in S(P)$ and $\pi = (1\,2) \in S(Q)$. Since,*

$$P_\sigma^{-1} \otimes P \oplus Q \otimes Q_\pi^{-1} = \begin{pmatrix} 0 & -3 \\ -2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -5 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 4 & 0 \end{pmatrix},$$

the digraph $\mathcal{G}(P_\sigma^{-1} \otimes P \oplus Q \otimes Q_\pi^{-1})$ has a circuit $(1, 2, 1)$ with weight 1. Thus, the strict inequality in (6) holds. Next, we consider another matrix,

$$R = \begin{pmatrix} 0 & 3 \\ 4 & 6 \end{pmatrix}.$$

For $\tau = (1\ 2) \in S(R)$, we compute,

$$P_\sigma^{-1} \otimes P \oplus R \otimes R_\tau^{-1} = \begin{pmatrix} 0 & -3 \\ -2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -4 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}.$$

Then, the digraph $\mathcal{G}(P_\sigma^{-1} \otimes P \oplus R \otimes R_\pi^{-1})$ has no positive weight circuit, and there is a vector,

$$\begin{pmatrix} -3 \\ 0 \end{pmatrix} \in W(P_\sigma^{-1} \otimes P) \cap W(R \otimes R_\tau^{-1}).$$

Thus, we have the equality $\det(P \otimes R) = \det P \otimes \det R$. The direct computation shows that,

$$\det(P \otimes R) = \det P \otimes \det R = 12.$$

The sets $W(P_\sigma^{\otimes -1} \otimes P), W(Q \otimes Q_\pi^{\otimes -1})$ and $W(R \otimes R_\tau^{-1})$ are illustrated in Figure 2.

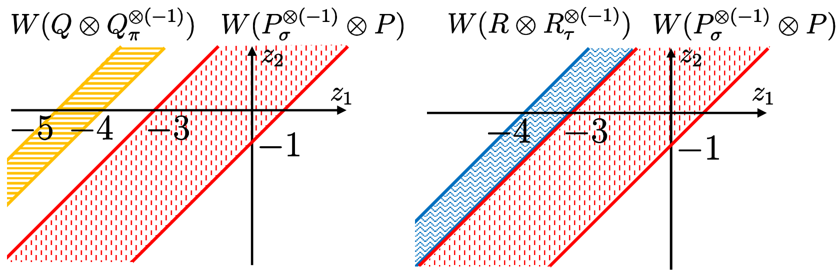

Figure 2: Illustration for $W(P_\sigma^{\otimes-1} \otimes P)$ and $W(Q \otimes Q_\pi^{\otimes-1})$ (left) and $W(R \otimes R_\tau^{-1})$ (right) in Example 3.2.

Example 3.3. *Let us consider,*

$$P = \begin{pmatrix} 0 & \varepsilon & 1 \\ \varepsilon & 0 & 2 \\ \varepsilon & \varepsilon & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \\ 1 & 2 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \\ -3 & -2 & 0 \end{pmatrix}.$$

We can easily verified that,

$$\det P = \det Q = \det R = 0,$$

and $S(P) = S(Q) = S(R) = \{\mathrm{id}\}$. *As*

$$P \oplus Q = \begin{pmatrix} 0 & \varepsilon & 1 \\ \varepsilon & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \quad P \oplus R = \begin{pmatrix} 0 & \varepsilon & 1 \\ \varepsilon & 0 & 2 \\ -3 & -2 & 0 \end{pmatrix},$$

the digraph $\mathcal{G}(P \oplus Q)$ *has a circuit* $(1, 3, 1)$ *with positive weight, while* $\mathcal{G}(P \oplus R)$ *has no circuit with positive weight. Hence, we have,*

$$\det(P \otimes Q) \neq \det P \otimes \det Q, \quad \det(P \otimes R) = \det P \otimes \det R.$$

In fact, the direct computation shows,

$$\det(P \otimes Q) = \det \begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \\ 1 & 2 & 0 \end{pmatrix} = 6, \quad \det(P \otimes R) = \det \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 2 \\ -3 & -2 & 0 \end{pmatrix} = 0.$$

We discuss the computational complexity for checking whether (7) holds for given $P, Q \in \mathbb{R}_{\max}^{n \times n}$ with m_1 and m_2 finite entries, respectively. Since the product of sparse matrices may not be sparse, the computation of $\det(P \otimes Q)$ requires $O(n^3)$ time in general. In contrast, by the Hungarian method [14], we can obtain $\sigma \in S(P)$ and $\pi \in S(Q)$ in $O(n(m_1 + n \log n))$ and $O(n(m_2 + n \log n))$ times of computations, respectively. As $P_\sigma^{-1} \otimes P \oplus Q \otimes Q_\pi^{-1}$ has at most $(m_1 + m_2)$ finite entries, it can be checked in $O(nm)$ time whether $\mathcal{G}(P_\sigma^{-1} \otimes P \oplus Q \otimes Q_\pi^{-1})$ has a positive circuit by the Karp algorithm [13], where $m = m_1 + m_2$. Thus, the validity of (7) can be checked in $O(n(m + n \log n))$ time, which is less than $O(n^3)$ when P and Q are sparse.

4 Conclusion

We discussed the determinant of a max-plus matrix from the viewpoint of the duality theorem. We derived a dual-type formula of max-plus algebraic determinant. Using that expression, we propose an equivalent condition for the multiplicative identity of max-plus algebraic determinant. As a possible future work, other max-plus algebraic objects might be characterized in terms of duality.

Acknowledgement This work is supported by JSPS KAKENHI Grant No. 22K13964.

Conflicts of Interest The authors declare no conflict of interest.

References

[1] M. Akian, S. Gaubert & A. Guterman (2014). Tropical Cramer determinants revisited. *Tropical and Idempotent Mathematics and Applications*, *616*, 1–45. https://doi.org/10.1090/conm/616.

[2] R. B. Bapat (1995). Permanents, max algebra and optimal assignment. *Linear Algebra and Its Applications*, *226*, 73–86. https://doi.org/10.1016/0024-3795(95)00304-A.

[3] R. Burkard, M. Dell'Amico & S. Martello (2012). *Assignment Problems: Revised Reprint*. Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania.

[4] P. Butkovič (2008). Finding a bounded mixed-integer solution to a system of dual network inequalities. *Operations Research Letters*, *36*(5), 623–627. https://doi.org/10.1016/j.orl.2008.04.004.

[5] P. Butkovič (1985). Necessary solvability conditions of systems of linear extremal equations. *Discrete Applied Mathematics*, *10*(1), 19–26. https://doi.org/10.1016/0166-218X(85)90056-3.

[6] P. Butkovič (2010). *Max-Linear Systems: Theory and Algorithms*. Springer Science & Business Media, London. https://doi.org/10.1007/978-1-84996-299-5.

[7] S. Gaubert, P. Butkovic & R. Cuninghame-Green (1998). Minimal $(\max, +)$ realization of convex sequences. *SIAM Journal on Control and Optimization*, *36*(1), 137–147. https://doi.org/10.1137/S036301299528534X.

[8] S. Gaubert & M. Plus (1997). Methods and applications of $(\max, +)$ linear algebra. In *Annual Symposium on Theoretical Aspects of Computer Science*, pp. 261–282. Springer, Lübeck, Germany. https://doi.org/10.1007/BFb0023465.

[9] M. Gondran & M. Minoux (2008). *Graphs, Dioids and Semirings: New Models and Algorithms*. Springer, New York, NY. https://doi.org/10.1007/978-0-387-75450-5.

[10] J. Hook, J. Pestana, F. Tisseur & J. Hogg (2019). Max-balanced Hungarian scalings. *SIAM Journal on Matrix Analysis and Applications*, *40*(1), 320–346. https://doi.org/10.1137/15M1024871.

[11] Z. Izhakian (2009). Tropical arithmetic and matrix algebra. *Communications in Algebra*, *37*(4), 1445–1468. https://doi.org/10.1080/00927870802466967.

[12] M. Joswig & G. Loho (2016). Weighted digraphs and tropical cones. *Linear Algebra and Its Applications*, *501*, 304–343. https://doi.org/10.1016/j.laa.2016.02.027.

[13] R. M. Karp (1978). A characterization of the minimum cycle mean in a digraph. *Discrete Mathematics*, *23*(3), 309–311. https://doi.org/10.1016/0012-365X(78)90011-0.

[14] H. W. Kuhn (1955). The Hungarian method for the assignment problem. *Naval Research Logistics Quarterly*, *2*(1-2), 83–97. https://doi.org/10.1002/nav.3800020109.

[15] G. J. Olsder & C. Roos (1988). Cramer and Cayley-Hamilton in the max algebra. *Linear Algebra and Its Applications*, *101*, 87–108. https://doi.org/10.1016/0024-3795(88)90145-0.

[16] A. Schrijver (2003). *Combinatorial Optimization: Polyhedra and Efficiency*. Springer, Berlin, Heidelberg.

[17] N. Tomizawa (1971). On some techniques useful for solution of transportation network problems. *Networks*, *1*(2), 173–194. https://doi.org/10.1002/net.3230010206.